

CANONICAL SPLITTINGS OF GROUPS AND 3-MANIFOLDS

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ABSTRACT. We introduce the notion of a ‘canonical’ splitting over $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ for a finitely generated group G . We show that when G happens to be the fundamental group of an orientable Haken manifold M with incompressible boundary, then the decomposition of the group naturally obtained from canonical splittings is closely related to the one given by the standard JSJ-decomposition of M . This leads to a new proof of Johannson’s Deformation Theorem.

An important chapter in the structure theory of Haken 3-manifolds is the theory of the characteristic submanifold initiated by Waldhausen in [28]. This was accomplished in 1974-75 by Jaco and Shalen [9] and Johannson [10]. They defined the characteristic submanifold $V(M)$ of a Haken 3-manifold M with incompressible boundary and proved it to be unique up to isotopy. It has the properties that each component is either a Seifert fibre space or is an I -bundle over a surface, and every essential map of an annulus or torus into M is properly homotopic into $V(M)$. Johannson called this last statement the Enclosing Property. This decomposition of M into submanifolds has come to be called the JSJ-decomposition. Johannson emphasised the importance of the JSJ-decomposition by using it to prove his Deformation Theorem [10], which gives a good description of the connection between Haken 3-manifolds that have isomorphic fundamental groups. More recently Sela [24] (see also Bowditch [1]) showed that analogous algebraic decompositions exist for word hyperbolic groups, and Rips and Sela extended aspects of this theory to torsion free finitely generated groups [17]. Dunwoody and Sageev [2] showed that such algebraic decompositions exist for all finitely presented groups, and Fujiwara and Papasoglu [5] further extended this theory. However, none of these group theoretic decompositions yields the usual JSJ-decomposition when restricted to fundamental groups of orientable Haken manifolds with incompressible boundary.

In this paper, we describe the standard JSJ-decomposition of an orientable Haken 3-manifold in terms that can be translated into group theory and we then give a purely algebraic description of it. This leads to a natural algebraic proof of Johannson’s Deformation Theorem [10]. The papers [16], [14], and [23] represent our earlier work in this programme. Our description of the standard JSJ-decomposition is the following. Instead of trying to characterize the characteristic submanifold, we concentrate on its frontier which consists of essential annuli and tori. Let M be an orientable Haken manifold with incompressible boundary. We call a two-sided embedded essential annulus or torus S in M *canonical* if any two-sided immersed

Received by the editors August 12, 2000 and, in revised form, April 9, 2001.

2000 *Mathematics Subject Classification.* Primary 57M07, 57N10, 20E06.

Key words and phrases. 3-manifold, characteristic submanifold, Deformation Theorem, ends of groups, intersection number, JSJ decomposition, splittings of groups.

The first author was partially supported by NSF grant DMS 034681.

essential annulus or torus T in M can be homotoped away from S . The Enclosing Property of the characteristic submanifold $V(M)$ of M implies that any component of the frontier of $V(M)$ is canonical. We will see in section 1 that, up to isotopy, these are the only canonical surfaces in M . Without assuming this, we can consider the collection of all isotopy classes of canonical surfaces in M and pick one surface from each isotopy class. Up to isotopy, any pair of these surfaces will be disjoint. By choosing a Riemannian metric on M and choosing each surface to be least area, it follows that we can choose all these surfaces to be disjoint. Thus we obtain a family of disjoint embedded essential annuli and tori in M such that no two of them are parallel. There is an upper bound to the number of such surfaces in M , which is called the Haken number of M . In particular, it follows that this must be a finite family, and we call it a JSJ-system for M . Such a system is unique up to isotopy. We will see in section 1 that a JSJ-system in M can be obtained from the frontier of the characteristic submanifold $V(M)$ of M by choosing one surface from each maximal collection of parallel surfaces. The reader is warned that in [16], Neumann and Swarup defined the term canonical in a slightly different way. They called S canonical if any *embedded* essential annulus or torus in M can be isotoped away from S . The two concepts are not equivalent. For example, let M be constructed by gluing two solid tori V and W along an annulus A which is embedded in their boundaries with degree greater than 1 in each. Thus M is a Seifert fibre space and A is an essential annulus in M . Now M contains no essential embedded tori and the only embedded essential annulus in M is A . Thus A is obviously canonical in the sense of Neumann and Swarup. But M does contain singular essential tori, and any such must intersect A , so that A is not canonical in our sense.

In order to describe an algebraic analogue of a JSJ-system, we need to have algebraic versions of the ideas of embedded surfaces, of immersed surfaces and of the property of being canonical. The analogue of a two-sided incompressible surface embedded in a 3-manifold is a splitting of a group along a surface group, which in our case will be a splitting of $\pi_1(M)$ along $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. The notion of a π_1 -injective immersed surface F in a 3-manifold M corresponds to an H -almost invariant subset of G , where G denotes $\pi_1(M)$ and H denotes the image of $\pi_1(F)$ in G . Such sets have been studied by Scott in [20] and [21], Dunwoody and others (see the references in [23]). The notion of homotopic disjointness of immersed surfaces can be formulated in terms of the intersection numbers developed by Freedman, Hass and Scott in [4]. It follows from that paper that two immersed surfaces can be homotoped to have disjoint images if and only if their intersection number equals zero. In [22], Scott showed that this notion of intersection number can be naturally extended to one for almost invariant sets. These ideas were further extended in [23] where we showed that almost invariant sets with self-intersection number zero naturally produce splittings, and that splittings with intersection number zero can be made “disjoint”. (In [23], the idea of disjointness of splittings is formulated precisely in terms of graphs of groups. See the end of section 1 in this paper.) Thus, we have all the notions needed to reformulate the above description of a JSJ-system in a 3-manifold M in a purely algebraic fashion.

One of the important aspects of the JSJ-decomposition for 3-manifolds is the description of the pieces obtained by splitting M along the JSJ-system. These are either simple or fibred or both and the characteristic submanifold $V(M)$ is essentially the union of the fibred pieces (see section 3). Since the Enclosing Property

implies that any immersed essential annulus or torus in M can be properly homotoped into $V(M)$, this may be taken as a description of all immersed essential annuli and tori in M , in particular, of all embedded essential annuli and tori. It is these aspects that the group theoretic generalizations (except the one for word hyperbolic groups) do not capture fully. Apart from I -bundle pieces, the characteristic submanifold consists of Seifert fibred pieces. The role played by those Seifert pieces which meet the boundary of M (we will call them *peripheral*) is rather different from that played by the other (interior) Seifert pieces. In all the group theoretic JSJ-decompositions, the peripheral Seifert fibred pieces are further decomposed, and the interior ones may or may not appear depending on which theory is being considered; but all the theories capture the analogues of the I -bundle pieces. The general difficulty seems to come from the fact that both annuli and tori are needed to describe the standard JSJ-decomposition whereas the group theoretic decompositions so far work well only with $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ splittings, but not both, and one type of intersection which we call strong crossing in [23]. Perhaps it may not be possible to fully capture all aspects of the standard JSJ-decomposition in the case of groups.

In section 1, we discuss background material on 3-manifolds and group theory. In section 2, we study the analogues of canonical annuli and tori, namely canonical $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ splittings of $G = \pi_1(M)$. If there is a true analogue for groups of the standard JSJ-decomposition for manifolds, one might hope that canonical annuli and tori correspond to canonical splittings. This turns out to be almost true. The exceptions are the canonical tori which have peripheral Seifert fibred spaces on both sides. However, we are also able to give a completely algebraic characterisation of these canonical tori. Since this characterisation is algebraic, it follows that the JSJ-systems of any two Haken manifolds with isomorphic fundamental groups correspond. In section 3, we use this fact to give a ‘natural’ algebraic proof of Johannson’s Deformation Theorem [10]. This result states that if M and N are Haken 3-manifolds with incompressible boundary, and $f : M \rightarrow N$ is a homotopy equivalence, then f is homotopic to a map g such that $g(V(M)) \subset V(N)$, the restriction of g to $\overline{V(M)}$ is a homotopy equivalence from $\overline{V(M)}$ to $V(N)$, and the restriction of g to $\overline{M - V(M)}$ is a homeomorphism onto $\overline{N - V(N)}$. Most of this result follows immediately from our algebraic description of the JSJ-decomposition. The proof of the last part of the Deformation Theorem which asserts the existence of a homeomorphism on the complement of $V(M)$ is carried out in section 3. This follows the ideas of Swarup [26] and Jaco [7], and corrects an omission in [7]. We remark that the proof in [26] is ‘algebraic’ and is one of the reasons for our algebraic point of view. The fact that topologically canonical tori need not correspond to algebraically canonical splittings indicates one of the difficulties in extending fully the standard JSJ-decomposition to groups. Since these exceptions have also an algebraic description, there seems to be some hope of carrying out some JSJ-decomposition for special classes of groups. The ideas here can also be used to obtain the decompositions of Sela [24] and Bowditch [1] for word hyperbolic groups.

1. BACKGROUND MATERIAL IN TOPOLOGY AND ALGEBRA

Recall that we say that a two-sided embedded essential annulus or torus S in M is canonical if any two-sided immersed essential annulus or torus T in M can be homotoped away from S . It will be convenient to reformulate the definition

of canonical in the following way, using the definition of intersection number of surfaces in a 3-manifold which was given in [4].

Definition 1.1. An essential annulus or torus embedded in M is *canonical* if it has intersection number zero with any (possibly singular) essential annulus or torus in M .

Recall that a JSJ-system in M is a family of disjoint embedded annuli and tori $S_1, \dots, S_n$ in M which contains one representative of each isotopy class of canonical annuli and tori. Such a system exists and is unique up to isotopy. We now describe the relationship between the JSJ-system in M and the characteristic submanifold $V(M)$ of M . Let $\mathcal{F}$ denote the family of annuli and tori in M obtained from the frontier $frV(M)$ of $V(M)$ by choosing one surface from each maximal collection of parallel surfaces.

Proposition 1.2. *Let M be an orientable Haken 3-manifold with incompressible boundary. Then $\mathcal{F}$ is a JSJ-system of M .*

Proof. The Enclosing Property of $V(M)$ shows that any component of $frV(M)$ is canonical. Thus $\mathcal{F}$ consists of non-parallel canonical surfaces and so is contained in a JSJ-system. Now any embedded essential annulus or torus can be isotoped to lie in $V(M)$. Thus the proof of the proposition reduces to showing that if a component X of $V(M)$ contains an embedded annulus or torus S which is essential in M , and is not parallel to a component of $frV(M)$, then S is not canonical. This means that there is some essential annulus or torus S' in M which has non-zero intersection number with S , or equivalently that S' cannot be properly homotoped apart from S . This can be proved by careful consideration of the possibilities using the fact that X is a Seifert fibre space or an I -bundle and that S can be isotoped to be vertical in X , i.e., to be a union of fibres in X . However, this consideration has essentially been done already in [16], so we will use their work to avoid a case by case discussion.

In [16], Neumann and Swarup defined what they called the W -system of M . This consisted of a maximal system of disjoint annuli and tori $S_1, \dots, S_n$ in M so that no two of the S_i are parallel, and each S_i is canonical in their sense, which means that any *embedded* essential annulus or torus in M can be isotoped away from each S_i . We would like to remark that the W -system is another natural concept that can be generalized to groups. The W -system is unique up to isotopy and they showed that the W -system contains $\mathcal{F}$. By the definition of the W -system, any essential annulus or torus disjoint from and not parallel into the W -system intersects some other embedded annulus or torus in an essential way and so does not belong to a JSJ-system. The only surfaces which are in the W -system but not in $\mathcal{F}$ are the special annuli described in Lemma 2.9 and Figure 5 of [16]. It is easy to see that in these cases (which essentially arise from Seifert bundles over twice punctured discs), there is an immersed torus intersecting a special annulus in an essential way. It follows that these special annuli also do not belong to the JSJ-system, so that the JSJ-system consists precisely of the W -system with these special annuli removed, which is the same as $\mathcal{F}$. $\square$

In this paper, we are restricting to the orientable case for simplicity, but it is easy to extend our analysis to the general case as follows. Let M be a non-orientable Haken 3-manifold, let M_1 denote its orientable double cover and consider

the canonical family $\mathcal{F}_1$ of annuli and tori in M_1 . In Theorem 8.6 of [12], it is shown that the tori in $\mathcal{F}_1$ can be chosen to be invariant under the covering involution unless M_1 is a bundle over the circle with fibre the torus and a hyperbolic gluing map. (Note that the full power of the results of [12] is not needed here. One need only know that any involution of $T \times I$ respects some product structure, and this was first proved by Kim and Tollefson [11].) Although it was not discussed in [12], the same argument shows that $\mathcal{F}_1$ itself can be chosen to be invariant. The only possible new exceptional case would be if M_1 is a bundle over the circle with fibre the annulus. But any such manifold is a Seifert fibre space, so that $\mathcal{F}_1$ is empty and is trivially invariant in this case. In particular, if M has non-empty boundary, we can always choose $\mathcal{F}_1$ to be invariant under the covering involution. The image of $\mathcal{F}_1$ in M is a collection of embedded essential annuli, tori, Moebius bands and Klein bottles, some of which may be one-sided. If one of these surfaces is one-sided, we replace it by the boundary of a thin regular neighbourhood. The resulting collection of two-sided surfaces in M is denoted by $\mathcal{F}$. As in the orientable case, the surfaces in $\mathcal{F}$ are characterised by the property that they are two-sided and essential, and have intersection number zero with every essential map of an annulus or torus into M .

We will need the following facts about the canonical decomposition of M obtained by cutting along the JSJ-system $\mathcal{F}$. The manifolds so obtained are called the canonical pieces of the decomposition. If a Seifert fibred canonical piece Σ contains canonical annuli in its frontier, then Σ admits a Seifert fibration for which all these canonical annuli are vertical in Σ . Note that if Σ is not closed, it has a unique Seifert fibration unless it is $S^1 \times D^2$, $T \times I$ or an I -bundle over the Klein bottle. We will need to give special consideration to any canonical torus T of M such that each of the canonical pieces of M adjacent to T is a Seifert fibre space which is peripheral, i.e., meets the boundary of M . We will say that such a canonical torus is *special*. Note that it is possible that a single Seifert fibred canonical piece of M meets both sides of a special canonical torus.

In [22], Scott defined intersection numbers for splittings of groups and more generally for almost invariant subsets of coset spaces of a group. See [22] and [23] for a detailed discussion of the connection of the algebraic idea of intersection number with the topological idea. In particular, let G denote $\pi_1(M)$, pick a basepoint in M and pick finitely many simple loops representing generators of G , which are disjoint except at the basepoint. Let Γ denote the pre-image of the union of these loops in the universal cover $\widetilde{M}$ of M . Once we have chosen a vertex of this graph as our basepoint in $\widetilde{M}$, we can identify the vertices of Γ with G , and can identify Γ itself with the Cayley graph of G with respect to the given generating set. If H is a subgroup of G , the quotient graph $H \backslash \Gamma$ is embedded in the quotient manifold $H \backslash \widetilde{M}$, which we will usually denote by M_H , and the vertices of $H \backslash \Gamma$ are naturally identified with the cosets Hg of H in G . Let F denote an annulus or torus and let $f : F \rightarrow M$ denote an essential map. This means that f is proper and π_1 -injective, and that f is not properly homotopic into the boundary ∂M of M . Let H denote the subgroup $f_*\pi_1(F)$ of G . Let M_F denote the cover of M such that $\pi_1(M_F) = H$, and consider the lift of f into M_F . For convenience, suppose that this lift is an embedding. (This is true up to homotopy and is automatically true if f is least area by [4].) Now let P and P^* denote the two submanifolds of M_F into which F cuts M_F . Thus the frontier in M_F of each of P and P^* is exactly F . As f is

essential, neither P nor P^* is compact. For if P were compact, it would have to be homeomorphic to $F \times I$ and so the lift of F would be properly homotopic into ∂M_F , contradicting our assumption that f is not properly homotopic into ∂M . We associate to P the subset of $H \backslash G$ consisting of those vertices of $H \backslash \Gamma$ which lie in the interior of P . Because the frontier of P is compact, this subset has finite coboundary in $H \backslash \Gamma$, and so is an almost invariant subset of $H \backslash G$. As neither P nor P^* is compact, this associated almost invariant set is non-trivial, which means that it is infinite with infinite complement. (Note that we are interested only in the almost equality class of this almost invariant subset, so it does not matter if some of the vertices of $H \backslash \Gamma$ lie on F .) For our purposes in this paper, we want to think of P as being much the same as its associated almost invariant set. However, in order to avoid confusion, we will not refer to P as an almost invariant set. Instead we make the following definitions.

Definition 1.3. A submanifold P (which need not be connected) of a 3-manifold W is an *ending submanifold* of W if P and the closure of $W - P$ are not compact, but the frontier of P is compact. Two ending submanifolds P and Q of W are *almost equal* if their symmetric difference $P - Q \cup Q - P$ is bounded.

Using the natural map which we have just defined from ending submanifolds of M_H to non-trivial almost invariant subsets of $H \backslash G$, leads to the following definition of the intersection number of two ending submanifolds.

Definition 1.4. Let M be a compact 3-manifold with fundamental group G , let H and K be subgroups of G , and let $P \subset M_H$ and $Q \subset M_K$ be ending submanifolds. Let X and Y denote the pre-images of P and Q respectively in the universal cover $\tilde{M}$ of M . Then Y crosses X if and only if each of the four sets $X \cap Y$, $X \cap Y^*$, $X^* \cap Y$, $X^* \cap Y^*$ has unbounded image in M_H . The *intersection number* of P and Q is the number of double cosets HgK such that gY crosses X .

It follows from [22] that crossing is symmetric, so that Y crosses X if and only if X crosses Y . Also the intersection number of P and Q is unaltered if either set is replaced by its complement or by an almost equal ending submanifold.

Our previous discussion shows how to associate an ending submanifold P of M_F to an essential annulus or torus in M . This submanifold is well defined up to complementation (we could equally well choose P^* in place of P), and up to almost equality. Note that P can be defined perfectly well even if the lift of f is not embedded. For the complement of the image of the lift we will have two unbounded components, and we choose one of them as P . We will need the following two definitions.

Definition 1.5. Given a splitting of G over H , where H is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, we will define this splitting to be *algebraically canonical* if the corresponding almost invariant subset of $H \backslash G$ has zero intersection number with any almost invariant subset of any $K \backslash G$, for which K is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

Definition 1.6. Given a splitting of G over H , where H is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, we will define this splitting to be *$\mathbb{Z}$ -canonical* if the corresponding almost invariant subset of $H \backslash G$ has zero intersection number with any almost invariant subset of any $K \backslash G$, for which K is isomorphic to $\mathbb{Z}$.

We have now defined what it means for an essential annulus or torus in M to be topologically canonical and for a splitting of G to be algebraically canonical or $\mathbb{Z}$ -canonical. In order to compare these algebraic and topological ideas, we reformulate them all in terms of ending submanifolds. Note that a splitting of $G = \pi_1(M)$ over H has an associated almost invariant subset of $H \backslash G$, and we can find an ending submanifold P of M_H whose associated almost invariant set is exactly this. We will say that P is associated to the splitting. Of course, P is not unique but any two choices for P will be almost equal or almost complementary. We have the following statements.

- Criterion 1.7.** (1) *A splitting of G over H , for which H is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}$, is algebraically canonical if and only if the associated ending submanifold P of M_H has intersection number zero with every ending submanifold Q of every M_K , for which K is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}$.*
- (2) *A splitting of G over H , for which H is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}$, is $\mathbb{Z}$ -canonical if and only if the associated ending submanifold P of M_H has intersection number zero with every ending submanifold Q of every M_K , for which K is isomorphic to $\mathbb{Z}$.*
- (3) *An essential annulus or torus F embedded in M , and carrying the group H , is topologically canonical if and only if the associated ending submanifold P of M_H has intersection number zero with every ending submanifold Q of every M_K , such that K is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}$ and Q is associated to an essential annulus or torus in M .*

Note that there is a natural topological idea corresponding to the algebraic idea of a splitting being $\mathbb{Z}$ -canonical. This is that an embedded essential torus or annulus in M is *annulus-canonical* if it has zero intersection number with every essential annulus in M . However, we will not need to use this idea in this paper.

Recall from the introduction that if we have a collection of canonical embedded annuli and tori in M , they can be isotoped to be disjoint. We discussed the algebraic analogue in [23], and will briefly describe the results proved there. Let $S_1, \dots, S_n$ be a family of disjoint essential embedded annuli or tori in a 3-manifold M with fundamental group G , so that each S_i determines a splitting of G . Together they determine a graph of groups structure on G with n edges. A collection of n splittings of a group G is said to be *compatible* if G can be expressed as the fundamental group of a graph of groups with n edges, such that, for each i , collapsing all edges but the i -th yields the i -th splitting of G . The splittings are *compatible up to conjugacy* if collapsing all edges but the i -th yields a splitting of G which is conjugate to the i -th given splitting. Clearly disjoint essential annuli and tori in M define splittings of G which are compatible up to conjugacy. The following result is Theorem 2.5 of [23].

Theorem 1.8. *Let G be a finitely generated group with n splittings over finitely generated subgroups. This collection of splittings is compatible up to conjugacy if and only if each pair of splittings has intersection number zero. Further, in this situation, the graph of groups structure on G has a unique underlying graph, and the edge and vertex groups are unique up to conjugacy.*

Now let G denote the fundamental group of a Haken 3-manifold, and consider a finite collection of algebraically canonical splittings of G over subgroups which are isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}$. Certainly any two such splittings have zero intersection

number, so together they yield a graph of groups structure for G such that, for each i , collapsing all edges but the i -th yields a splitting of G which is conjugate to the i -th given splitting. Further this graph is unique.

2. THE MAIN RESULTS

The aim of this section is to relate the topological and algebraic ideas of being canonical which we have introduced in the previous section.

Our major result is the following.

Theorem 2.1. *Let M be an orientable Haken 3-manifold with incompressible boundary, and let Φ denote the natural map from the set of isotopy classes of embedded essential annuli and tori in M to the set of conjugacy classes of splittings of $G = \pi_1(M)$ over a subgroup isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}$.*

- (1) *If σ is an algebraically canonical splitting of G over $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, then there is a canonical annulus or torus F in M , such that $\Phi(F) = \sigma$.*
- (2) *If F is a canonical annulus or torus in M , then either $\Phi(F)$ is an algebraically canonical splitting of G over $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$, or F is a special canonical torus in M .*

Remark 2.2. We will show in Lemma 2.12 that if F is a special canonical torus in M , then $\Phi(F)$ is never algebraically canonical. Thus Φ induces a bijection between the collection of isotopy classes of non-special canonical annuli and tori in M and the collection of conjugacy classes of algebraically canonical splittings of G . (Recall that, by definition, the word special only applies to tori.)

We prove part (1) of Theorem 2.1 in Lemmas 2.6 and 2.9, and we prove part (2) in Lemmas 2.10 and 2.11.

Note that if a canonical splitting of G is induced by an embedded annulus or torus F , it is obvious that F is topologically canonical, by Criterion 1.7. The difficulty is that many splittings of G over $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ are not induced by a splitting over a connected surface. For a very simple example of this phenomenon in one dimension less, let Σ denote the disc with two holes, i.e., the compact planar surface with three boundary components. There are only six essential arcs in Σ each of which defines a splitting of $G = \pi_1(\Sigma)$ over the trivial group. (Three of these arcs join distinct components of $\partial\Sigma$, and the other three do not.) But there are infinitely many non-conjugate splittings of the free group of rank two over the trivial group, corresponding to the infinitely many choices of generators. Thus all of these splittings except for six are not induced by a connected 1-manifold in Σ . For a 3-dimensional example, we simply take the product of Σ with the circle S^1 . This will yield a compact 3-manifold $\Sigma \times S^1$ for which infinitely many splittings over $\mathbb{Z}$ are not represented by a connected surface.

To start the proof, we consider general ending submanifolds in a cover M_H of M , with H infinite cyclic. We know that the boundary of M_H is incompressible, and so consists of planes and open annuli. If A_1 and A_2 are annulus components of ∂M_H , there is an essential loop on A_1 homotopic to a loop on A_2 . The Annulus Theorem [20] then implies that there is an embedded incompressible annulus in M_H joining A_1 and A_2 . It follows that all the annulus components of ∂M_H carry the same subgroup of H , and hence so does any properly embedded compact incompressible annulus in M_H . It can be shown that M_H compactifies to a solid torus, but we will not need to use this fact.

Lemma 2.3. *Any ending submanifold P of M_H , for which H is isomorphic to $\mathbb{Z}$, is almost equal to one with the property that the frontier dP of P consists only of embedded essential annuli, and distinct boundary circles of dP lie in distinct boundary components of M_H .*

Proof. Perform surgery on the frontier dP of P to make it π_1 -injective in M_H . This adds to P or subtracts from P compact sets, namely 2-handles attached to dP , so that, by changing P by a compact set, we can arrange that P has π_1 -injective frontier dP . As $\pi_1(M_H) = H$ is isomorphic to $\mathbb{Z}$, the components of dP can only be annuli, discs or spheres. As M_H is irreducible, any sphere bounds a ball, so that any spheres in dP can be removed by adding this ball to P or removing it, as appropriate. As M_H has incompressible boundary, if D is a disc properly embedded in M_H with boundary C , then C bounds a disc D' in ∂M_H . Thus $D \cup D'$ is a sphere in M_H and so bounds a ball. As before, any discs in dP can be removed by adding this ball to P or removing it, as appropriate. Similarly, inessential annuli can be removed from dP . Thus we eventually arrive at a stage where dP consists of a finite number of compact essential annuli in M_H . If two boundary circles of dP lie in a single component S of ∂M_K , they will cobound a compact annulus A in S , and we can alter P by adding or removing a regular neighbourhood of A . This reduces the number of boundary circles of dP , so by repeating, we can ensure that distinct boundary circles of dP lie in distinct boundary components of M_K as required. (This move may produce an inessential annulus component of dP , in which case we can remove it as above.) This completes the proof of the lemma. $\square$

The following technical result will be extremely useful in what follows.

Lemma 2.4. *Let $A_1, \dots, A_k$ be disjoint compact incompressible annuli in M_H such that distinct boundary circles of the A_i 's lie in distinct boundary components of M_H . Let R denote the closure of a component of $M_H - \bigcup_{i=1}^k A_i$, and let A denote an embedded incompressible annulus in R which joins A_1 to A_2 . Then A cuts R into two unbounded components.*

Proof. Recall that H is infinite cyclic. This implies that A carries a subgroup of finite index in $\pi_1(R)$, so that A must separate R into two pieces V and W . Now let $S_1, \dots, S_4$ denote the four components of ∂M_H which contain the four boundary components of A_1 and A_2 . Thus each S_i is an open annulus and A_1 and A_2 cut each S_i into two unbounded annuli. As each S_i contains no other boundary component of any A_j , this means that each of V and W contains two of these unbounded annuli, and so must be unbounded. This completes the proof of the lemma. $\square$

Lemma 2.5. *Suppose that P is an ending submanifold of M_H , for which H is isomorphic to $\mathbb{Z}$, such that P is associated to a canonical splitting of G over H . Then P is almost equal to an ending submanifold whose frontier consists of a single essential annulus which carries H .*

Proof. Lemma 2.3 tells us that we can arrange that dP consists only of embedded essential annuli, and that distinct boundary circles of dP lie in distinct boundary components of M_H .

First we will show that dP is connected. Otherwise, pick two components of dP . There is a compact incompressible annulus A embedded in M_H and joining these two components. It is possible that A meets other components of dP in its interior, but we can isotop A so that it meets each component of dP in essential

circles only. Thus A has a sub-annulus which joins distinct components A_1 and A_2 of dP and whose interior does not meet dP . We will replace A by this sub-annulus and will continue to call this annulus A . Without loss of generality, we can suppose that A lies in a component Q of P^* , and cuts Q into pieces V and W . Lemma 2.4 shows that V and W must each be unbounded. Let P_1 and P_2 denote the components of P which contain A_1 and A_2 respectively and let R denote the submanifold $V \cup P_1$ of M_H . Then all four of the intersection sets of R and R^* with P and P^* are unbounded, so we have found an ending submanifold of M_H with non-zero intersection number with the ending submanifold P , contradicting our hypothesis that P is associated to a canonical splitting of G over H . This contradiction shows that dP must be connected.

Next we show that dP must carry H . Otherwise, dP carries a proper subgroup K of finite index in H . Let Q denote the pre-image of P in M_K , so that dQ is the pre-image of dP in M_K and consists of at least two embedded essential annuli. Note that distinct boundary circles of dQ must lie in distinct boundary components of M_K . Now we can apply the argument of the preceding paragraph with M_K in place of M_H and Q in place of P . This will yield an ending submanifold of M_K with non-zero intersection number with the ending submanifold Q and hence with P . This contradicts our hypothesis that P is associated to a canonical splitting of G over H , completing the proof of the lemma. $\square$

Lemma 2.6. *An algebraically canonical splitting of $G = \pi_1(M)$ over H , which is isomorphic to $\mathbb{Z}$, is induced by a topologically canonical embedded essential annulus in M .*

Proof. Lemma 2.5 tells us that if we start with such a splitting of G over H , the associated ending submanifold P of M_H can be chosen to have frontier consisting of a single essential annulus A which carries H . We will show that A can be chosen so that its projection into M is an embedding whose image we again denote by A . Thus A induces the given splitting of G over H . As pointed out earlier, it is trivial that A must be topologically canonical.

In order to prove that we can choose A so as to embed in M , it will be convenient to choose A to be least area in its proper homotopy class. It is shown in [4] that any such least area annulus will also be embedded in M_H . (There are two options here. One can choose a Riemannian metric on M and then choose A to minimise smooth area (see [13] or [15]) or one can follow the ideas of Jaco and Rubinstein [8] and choose a hyperbolic metric on the 2-skeleton of some fixed triangulation of M and choose A to minimise PL -area. It does not matter which choice is made.) The fact that P has zero intersection number with any ending submanifold of any M_K , with K isomorphic to $\mathbb{Z}$, implies, in particular, that the pre-image X of P in $\widehat{M}$ does not cross any of its translates. As the frontier dX of X is an infinite strip covering A and so is connected, it follows that dX crosses none of its translates. This means that, for all $g \in G$, at least one of $gdX \cap X$ and $gdX \cap X^*$ has bounded image in M_H . In turn, this implies that the self-intersection number of A is zero so that A must cover an embedded surface in M ; see [4]. If A properly covers an embedded annulus, then the stabiliser of X will strictly contain H . But as P is associated to a splitting of G over H , we know that the stabiliser of X is exactly H . It follows that either A embeds in M as required, or that it double covers an embedded 1-sided Moebius band B in M . In the second case, we can isotope A slightly so that it embeds in M as the boundary of a regular neighbourhood of B .

Thus in all cases, we can choose A so that its projection into M is an embedding as required.

This completes the proof that the given algebraically canonical splitting of G comes from a topologically canonical annulus. $\square$

Next we go through much the same argument with the subgroup H of G replaced by a subgroup K which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. As before, we start with a general discussion of ending submanifolds. Consider the cover M_K of M with $\pi_1(M_K) = K$. The boundary of M_K is incompressible, and so consists of planes, annuli and possibly a single torus. An embedded incompressible torus T in the interior of M_K splits it into two pieces which we will refer to as the left and the right, and the boundary components of M_K are correspondingly split into left and right boundary components. Let A_1 and A_2 be distinct left annulus components of ∂M_K . As any loop on a left annulus component of ∂M_K is homotopic into T , the Annulus Theorem yields an embedded incompressible annulus B_i joining A_i to T , $i = 1, 2$. As A_1 and A_2 are distinct, it is trivial that the boundary components of B_1 and B_2 which do not lie in T are disjoint. Now it is easy to show that B_1 and B_2 can be isotoped to be disjoint. (Their intersection must consist of circles and of arcs with both endpoints on T . All nullhomotopic circles can be removed by an isotopy, and then all such arcs can be removed by an isotopy, starting with innermost arcs. Finally, one can remove any essential circles by an isotopy.) This implies that all left annulus components of ∂M_K carry the same subgroup of K , and the same comment applies to right annulus components of ∂M_K . If P is an ending submanifold of M_K , an annulus component of dP will be called a left annulus if each of its boundary circles lies in a left annulus boundary component of M_K . Right annulus components of dP are defined similarly. Note that an annulus component of dP may have one boundary circle in a left annulus component of ∂M_K and the other in a right annulus component, but this can only happen when the left and right annulus components of ∂M_K all carry the same group. We will call such an annulus *mixed*. It will not separate M_K , as it can be isotoped to intersect T in a single essential circle C and then T contains a circle which meets C transversely in a single point. But any other properly embedded compact incompressible annulus A in M_K must separate M_K . This is because A can be isotoped to be disjoint from T , which implies that one component of $M_K - A$ has fundamental group equal to the fundamental group of M_K . It can be shown that M_K compactifies to $T \times I$, but we will not need this fact.

Lemma 2.7. *Any ending submanifold P of M_K , for which K is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, is almost equal to one with the following properties:*

- (1) *The frontier consists only of embedded essential annuli or tori, such that distinct boundary circles lie in distinct boundary components of M_K .*
- (2) *The frontier has at most one torus component.*

Proof. Let P be an ending submanifold of M_K . As in the proof of Lemma 2.3, we can arrange that dP consists of essential annuli and tori, and that distinct boundary circles of dP lie in distinct boundary components of M_K . We can also arrange that dP has at most one torus component, as any two incompressible tori in M_K cobound a product region in M_K , which can be added to P or subtracted from P , as appropriate. This completes the proof of the lemma. $\square$

Lemma 2.8. *Suppose that P is an ending submanifold of M_K , for which K is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, such that P is associated to a canonical splitting of G over K . Then P is almost equal to a connected ending submanifold whose frontier consists of a single essential torus.*

Proof. We can assume that P is chosen as in Lemma 2.7 above.

First we show that dP cannot have an annulus component which is mixed. For suppose that dP has an annulus component joining a left annulus component A_L and a right annulus component A_R of ∂M_K . As ∂M_K has left and right annulus components, it does not have a torus component so there is an essential torus in M_K . This torus is the frontier of an ending submanifold R of M_K , and we consider the four intersections $P^{(*)} \cap R^{(*)}$. None of these can be compact as each contains a non-compact piece of A_L or of A_R in its boundary. This uses the fact that distinct boundary circles of dP lie in distinct components of ∂M_K . It follows that P and R have non-zero intersection number contradicting our hypothesis that P is associated to a canonical splitting of G over K . This contradiction shows that dP does not have a mixed annulus component. Note that this implies that each component of dP separates M_K .

Second we will show that dP cannot have any components which are left or right annuli. The only remaining possibility will then be that dP consists of a single torus as claimed. Note that in the following proof, we will only use the fact that the given splitting is $\mathbb{Z}$ -canonical, but not the fact that it is algebraically canonical. This will allow us to use the same argument in the last part of the proof of Theorem 2.14.

Suppose that dP has a left annulus component A which carries the subgroup H of K . Then the pre-image of A in M_H consists of infinitely many copies of A . Let Q denote the full pre-image of P in M_H . Of course, Q is not an ending submanifold of M_H as its frontier dQ is not compact. The components of dQ may be compact annuli, infinite strips or an open annulus if dP has a torus component. A compact annulus component of dQ will be called left or right depending on whether it projects to a left or right annulus component of dP . Recall that distinct boundary circles of dP lie in distinct boundary components of M_K . It follows that distinct boundary circles of dQ lie in distinct boundary components of M_H . As dQ contains more than one left annulus component, there is an incompressible annulus B embedded in M_H and joining two such components.

It is possible that B meets other components of dQ in its interior, but we can isotop B so that it meets each component of dQ in essential circles only. Hence, if B meets a component C of dQ , then C must be an annulus, compact or open, and must separate M_H . It follows that if B meets C in more than one essential circle, we can alter B so as to remove two circles of intersection with C , essentially by replacing a sub-annulus of B by a sub-annulus of C . As the boundary circles of B lie in left annulus components of dQ , we can arrange, by repeating this argument, that B meets only left annulus components of dQ . Now B must have a sub-annulus which joins distinct left annulus components A_1 and A_2 of dQ and whose interior does not meet dQ . We will replace B by this sub-annulus and will continue to call this annulus B .

Without loss of generality, we can assume that B lies in a component R of Q^* , and we let Q_1 and Q_2 denote the components of Q which contain A_1 and A_2 respectively. This annulus B separates R into two pieces U and V , and the argument of Lemma 2.4 shows that neither U nor V can be compact. Recall that if

A is any left annulus component of dP in M_K , then A carries the group H and cuts M_K into two pieces one of which also carries H . It follows that any left annulus component of dQ cuts M_H into two pieces one of which projects into M_K by a homeomorphism. As A_1 and A_2 are such annuli, it follows that at least one of Q_1 and Q_2 projects into M_K by a homeomorphism. In particular, at least one of Q_1 and Q_2 is an ending submanifold of M_H . Assume that Q_1 is an ending submanifold. Cut M_H along $A_1 \cup B \cup A_2$, and let W denote the piece so obtained which contains X . Thus W is also an ending submanifold. If W^* contains no translate of Q_1 other than Q_1 itself, we remove one such translate from W to obtain a new ending submanifold Z such that Z^* contains at least two translates of Q_1 . Otherwise we let Z equal W . Now we let $S = Z \cup Q_1$ and note that S is an ending submanifold of M_H and that S^* contains a translate of Q_1 . This implies that all four of the sets $S^{(*)} \cap Q^{(*)}$ are unbounded. If X and Y denote the pre-images of P and S respectively in $\widetilde{M}$, it follows that each of the four sets $X^{(*)} \cap Y^{(*)}$ has projection into M_H which is unbounded, so that S and P have non-zero intersection number. This contradicts the assumption that P is associated to a canonical splitting of G over K , so we conclude that dP cannot have a component which is a left annulus. Similarly, dP cannot have a component which is a right annulus. This completes the proof of the lemma. $\square$

Lemma 2.9. *An algebraically canonical splitting of $G = \pi_1(M)$ over K , which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, is induced by a topologically canonical embedded essential torus in M .*

Proof. Lemma 2.8 tells us that if we start with such a splitting of G over K , the associated ending submanifold P of M_K can be chosen to have frontier consisting of a single essential torus T . It is automatic that T carries K . We will show that T can be chosen so that its projection into M is an embedding whose image we again denote by T . Now T induces the given splitting of G over K . As pointed out earlier, it is trivial that T must be topologically canonical.

In order to prove that we can choose T so as to embed in M , it will again be convenient to choose T to be least area. See [18] or [8] for the existence results in the smooth and PL cases respectively. Again [4] shows that a least area torus is embedded. The fact that P has zero intersection number with any ending submanifold of any M_K , with K isomorphic to $\mathbb{Z} \times \mathbb{Z}$, implies in particular that the pre-image X of P in $\widetilde{M}$ does not cross any of its translates. Hence the frontier dX of X , which is a plane above T , crosses none of its translates. This means that, for all $g \in G$, at least one of $gdX \cap X$ and $gdX \cap X^*$ has bounded image in M_K . In turn, this implies that the self-intersection number of T is zero so that T must cover an embedded surface in M ; see [4]. If T covers an embedded torus, then the stabiliser of X will contain K . But as P is associated to a splitting of G over K , we know that the stabiliser of X is exactly K . It follows that either T embeds in M as required, or that it double covers an embedded 1-sided Klein bottle in M . In this case, we can isotop T slightly so that it embeds in M as the boundary of a regular neighbourhood of the Klein bottle. Thus in all cases, we can choose T so that its projection into M is an embedding as required.

This completes the proof that the given algebraically canonical splitting of G comes from a topologically canonical torus. $\square$

At this point we have proved part 1 of Theorem 2.1 by showing that a canonical splitting of G over $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$ comes from a canonical annulus or torus in M . These results were proved in Lemmas 2.6 and 2.9. Now we need to show that a non-special canonical annulus or torus in M induces a canonical splitting of G .

Lemma 2.10. *Let A be a topologically canonical annulus in M . Then A determines an algebraically canonical splitting of $G = \pi_1(M)$.*

Proof. Let H denote the group carried by A , so that we have a splitting of G over H . Lift A into M_H , and let P denote one side of A in M_H . Let X denote the pre-image in $\widetilde{M}$ of P . We consider the annulus boundary components of M_H . There are two which contain the boundary circles of A . The others lie on the left or right of A . As all these annuli carry exactly the same group, namely H , we cannot have such annuli on both sides of A , as this would immediately yield a compact annulus in M_H crossing A in an essential way, contradicting our hypothesis that A is topologically canonical. Hence we can suppose that P contains no annulus component of ∂M_H .

Let K be a subgroup of G isomorphic to $\mathbb{Z}$, and let Q be an ending submanifold of M_K with pre-image Y in $\widetilde{M}$. We will show that X and Y do not cross. If this is known for every such K and Q , it will follow that P has intersection number zero with every such Q , as required. Lemma 2.3 tells us that we can choose Q to be a submanifold of M_K with frontier consisting only of essential annuli, and all these annuli carry the same group. Thus Y has frontier consisting of finitely many infinite strips, which are “parallel” in the sense that each lies within a uniformly bounded distance of each other. We consider how these strips project into M_H . If one (and hence every) such strip has compact image in M_H , that image will be a (possibly singular) annulus properly mapped into M_H . As P contains no annulus component of ∂M_H , it follows that each of these annuli can be properly homotoped to lie in P^* . After this homotopy, one of $Y \cap P$ or $Y^* \cap P$ would be empty. It follows that, before the homotopy, one of these two sets projected to a bounded subset of M_H , so that P and Q do not cross, as required. Otherwise, the image of each of these strips in M_H is non-compact and the map into M_H is proper. Thus each end of each strip is mapped to the left or to the right side of A . If one of these strips has its two ends on opposite sides of A , the corresponding annulus component of dQ has non-zero intersection number with A , contradicting our hypothesis that A is topologically canonical. It follows that each of these strips has both of its ends on the same side of A , and hence that all the ends of all the strips are on one side of A . Hence they can be properly homotoped so as to be disjoint from A , which again shows that X and Y do not cross as required.

Now let K be a subgroup of G isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and let Q be an ending submanifold of M_K with pre-image Y in $\widetilde{M}$. We will show that X and Y do not cross. This will show that P has intersection number zero with every such Q . Lemma 2.7 tells us that we can choose Q so that dQ consists of essential annuli and possibly a torus in M_K . Thus Y has frontier dY consisting of infinite strips and possibly a plane. In any case, if T denotes an incompressible torus in M_K , it yields a plane Π in $\widetilde{M}$ such that dY lies in a bounded neighbourhood of Π . Now we consider how Π projects into M_H . There are two cases. Either Π projects into M_H to yield a properly immersed plane, or Π projects to a properly immersed annulus. In the first case, as a plane has only one end, and as a properly immersed plane in M_H can only meet A in a compact set, it follows that the intersection of Π with X

or X^* is compact. As dY lies in a bounded neighbourhood of Π , it follows that dY meets X or X^* in a compact set. This implies that one of the four intersection sets $X^{(*)} \cap Y^{(*)}$ is bounded, so that X and Y do not cross as required. In the second case, there are two subcases depending on whether the two ends of the annulus image of Π are mapped to the same or opposite sides of A . If they are mapped to the same side, then much as above, it follows that one of the four intersection sets $X^{(*)} \cap Y^{(*)}$ has bounded image in M_H , so that X and Y do not cross as required. If they are mapped to opposite sides of A , it follows that the torus T in M_K has non-zero intersection number with A , which contradicts our hypothesis that A is topologically canonical. This concludes the proof that a topologically canonical annulus in M determines an algebraically canonical splitting of G . $\square$

Next we prove the corresponding result for a torus. The case of special canonical tori in M will be discussed at the end of this section.

Lemma 2.11. *Let T be a topologically canonical torus in M , which is not special. Then T determines an algebraically canonical splitting of $G = \pi_1(M)$.*

Proof. Let H denote the group carried by T , so that we have a splitting of G over H . Lift T into M_H , and let P denote one side of T in M_H . Let X denote the pre-image in $\widetilde{M}$ of P . As ∂M_H cannot have a torus component, it must consist of a (possibly empty) collection of left annuli and right annuli and planes. If there are any left annulus components, it follows that the canonical component of M containing the left side of T is a peripheral Seifert fibre space. Similarly for right annuli. As we are assuming that T is not special, it follows that ∂M_H cannot have both left annulus and right annulus components. Without loss of generality, suppose that ∂M_H has no left annulus components.

Let K be a subgroup of G isomorphic to $\mathbb{Z}$, and let Q be an ending submanifold of M_K with pre-image Y in $\widetilde{M}$. We will show that X and Y do not cross. As before, this will show that P has intersection number zero with every such Q . Lemma 2.3 tells us that we can choose Q to be a submanifold of M_K with frontier consisting only of essential annuli, and all these annuli carry the same group. Thus the frontier dY of Y consists of finitely many ‘parallel’ infinite strips. As before there are two cases depending on whether these strips project properly into M_H or project to compact annuli in M_H .

Suppose that dY projects to compact annuli in M_H . The boundary lines of dY then project to circles in ∂M_H which must all lie on the right side of T , as ∂M_H has no left annulus components. Thus we can homotop dY to arrange that dY lies entirely on one side of T . It follows that before the homotopy, one of the four intersection sets $X^{(*)} \cap Y^{(*)}$ has bounded image in M_H , so that X and Y do not cross as required.

Next suppose that dY projects to infinite strips in M_H . Each boundary line of each such strip must lie in a boundary component of M_H . As the intersection of a strip with T is compact, the two boundary lines of a single strip must lie on the same side of T , so that the strip can be homotoped to be disjoint from T . As the infinite strips forming dY are parallel, it follows that they all lie on the same side of T , and again X and Y do not cross.

Now let K be a subgroup of G isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and let Q be an ending submanifold of M_K with pre-image Y in $\widetilde{M}$. We will show that X and Y do not cross. As before, this will show that P has intersection number zero with every

such Q . Lemma 2.7 tells us that we can assume that dQ consists of essential annuli and possibly one torus. Thus dY consists of infinite strips and possibly a plane. As in the proof of Lemma 2.10, we consider the projection of dY into M_H . An infinite strip component of dY must project to a properly immersed infinite strip or to an annulus. As above, in either case, the image of any one strip component of dY can be homotoped to lie on one side of T . If the image of a strip is a singular annulus, its boundary curves can only lie on the right side of T .

Let S denote an incompressible torus in M_K , and let Π denote the pre-image plane in $\tilde{M}$. Thus dY lies in a bounded neighbourhood of Π . There are three cases for the projection of Π into M_H . We can obtain a proper map of a plane, torus or annulus.

If Π is mapped properly into M_H , the fact that Π has only one end implies that all except a compact subset of Π maps to one side of T . Also the infinite strips in dY must all project to infinite strips each within a bounded neighbourhood of the image of Π . It follows that we can homotop dY to lie on one side of T , so that X and Y do not cross.

If the image of Π in M_H is a singular torus, it follows that every infinite strip of dY has image a singular annulus and so has boundary lying on the right side of T . Thus dY can be homotoped into the right side of T , and again X and Y do not cross.

If the image of Π in M_H is a singular annulus, it will be convenient to let Σ_L and Σ_R denote the canonical pieces of M which meet the left and right sides of T respectively. Of course, it is possible that $\Sigma_L = \Sigma_R$. Now the image of Π is a singular annulus whose two ends must lie on the same side of T , as otherwise the tori S and T would intersect essentially, contradicting our hypothesis that T is a canonical torus. If an infinite strip in dY has image an infinite strip in M_H , that strip must again have its boundary on the same side of T as the ends of the singular annulus which is the image of Π . If an infinite strip in dY has image a singular annulus, we know that its boundary must lie on the right side of T and so can be homotoped to lie entirely on the right side of T . Thus either all of dY can be homotoped into one side of T , so that X and Y do not cross, or dY has some infinite strips which project to singular annuli whose boundary curves lie on the right of T , but the ends of the image of Π lie on the left of T . We will show that this last case cannot occur. Suppose it does occur. Then S projects to a torus in M which is homotopic into Σ_L . As the image of Π in M_H is an annulus, S and T cannot be homotopic, so it follows that Σ_L is a Seifert fibre space. Also there is a singular compact annulus A in M_H which has one boundary component on S , meets T transversely in a single circle, and has its second boundary component in an annulus boundary component of ∂M_H . The existence of A implies that Σ_R is a Seifert fibre space. Further the existence of A shows that Σ_L and Σ_R can be fibred to induce the same fibration on T . This contradicts the fact that T is a canonical torus in M , so we deduce that this last possibility cannot occur.

This completes the proof that a topologically canonical torus in M , which is not special, determines an algebraically canonical splitting of G . $\square$

At this point, we have completed the proof of part (2) of Theorem 2.1 by proving Lemmas 2.10 and 2.11. To complete this section, we need to discuss the algebraic analogue of special canonical tori in M . First we show that these tori never define algebraically canonical splittings.

Lemma 2.12. *If T is a special canonical torus in M , then $\Phi(T)$ is not algebraically canonical.*

Proof. Let T be a special canonical torus in M , and let H denote the group carried by T , so that we have a splitting of G over H . Lift T into M_H , and let P denote one side of T in M_H . Let X denote the pre-image in $\widetilde{M}$ of P . The hypotheses imply that ∂M_H has at least one annulus component on each side of T . As before we will refer to the two sides of T as the left and right.

Suppose first that ∂M_H has at least two annulus components on each side of T . Pick a compact left-annulus A_L in M_H joining two distinct left annulus components of ∂M_H and let A_R be a similarly defined right annulus. Let Q_L denote the component of $M_H - A_L$ which does not contain T , let Q_R denote the component of $M_H - A_R$ which does not contain T , and let $Q = Q_L \cup Q_R$. Then clearly all four of the intersections $P^{(*)} \cap Q^{(*)}$ are unbounded, so that P and Q have non-zero intersection number and so $\Phi(T)$ is not algebraically canonical in this case.

If ∂M_H has only one annulus component on one side of T , (or on both sides of T), we simply make the preceding argument in a double cover of M_H chosen so that each annulus component of ∂M_H has two components above it in the double cover. $\square$

In the above proof, we exhibited an ending submanifold of M_H (or of a double cover of M_H) which has non-zero intersection number with the ending submanifold P of M_H determined by T . This ending submanifold need not correspond to a splitting of G , but the following example shows that one can often choose it to correspond to a splitting.

Example 2.13. Let F_1 and F_2 denote two compact surfaces each with at least two boundary components. Let Σ_i denote $F_i \times S^1$, let T_i denote a boundary component of Σ_i , and construct a 3-manifold M from Σ_1 and Σ_2 by gluing T_1 to T_2 . Let T denote the torus $\Sigma_1 \cap \Sigma_2$, and let H denote the subgroup of $G = \pi_1(M)$ carried by T . Then there is a splitting of G over H which has non-zero intersection number with $\Phi(T)$. If T_1 is glued to T_2 so that the given fibrations by circles do not match, then T is a special canonical torus in M . (If the fibrations are matched, then M is itself a Seifert fibre space and so has no canonical tori.)

Let G_i denote $\pi_1(\Sigma_i)$, and let C_i denote the subgroup of G_i carried by T_i . The starting point of our construction is that if F is a compact surface with at least two boundary components, and if S denotes a boundary circle of F , then S carries an infinite cyclic subgroup of $\pi_1(F)$ which is a free factor of $\pi_1(F)$. Now it is easy to give a splitting of $\pi_1(F)$ over $\pi_1(S)$, and hence a splitting of G_i over C_i . If each $\pi_1(F_i)$ is free of rank at least 3, then we can write $G_i = A_i *_{C_i} B_i$. If we let A denote the subgroup of G generated by A_1 and A_2 , i.e., $A = A_1 *_H A_2$, and define B similarly, then we can express G as $A *_H B$, and it is easy to see that this splitting of G has non-zero intersection number with $\Phi(T)$. If $\pi_1(F_i)$ has rank 2, then $G_i = A_i *_{C_i}$ and a similar construction can be made.

Our next result gives an algebraic characterisation of special canonical tori.

Theorem 2.14. (1) *If T is a special canonical torus in M , carrying the group H , then $\Phi(T)$ is $\mathbb{Z}$ -canonical, and G has splittings over two incommensurable infinite cyclic subgroups of H .*

(2) *If σ is a $\mathbb{Z}$ -canonical splitting of G over a subgroup H which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, such that G has splittings over two incommensurable infinite cyclic*

subgroups of H , then there is a special canonical torus T in M , such that $\Phi(T) = \sigma$.

Proof. 1) Lift T into M_H , and let P denote one side of T in M_H . Let X denote the pre-image in $\widetilde{M}$ of P . As T is special, the canonical pieces of M on each side of T are Seifert fibre spaces which meet ∂M either in tori or in vertical annuli. Hence ∂M_H consists of a non-empty collection of left annuli and right annuli and of planes, with the left annuli and right annuli carrying incommensurable subgroups of H .

Let K be a subgroup of G isomorphic to $\mathbb{Z}$, and let Q be an ending submanifold of M_K with pre-image Y in $\widetilde{M}$. We will show that X and Y do not cross. As before, this will show that P has intersection number zero with every such Q . Lemma 2.3 tells us that we can choose Q to be a submanifold of M_K with frontier consisting only of essential annuli, and all these annuli carry the same group. Thus the frontier dY of Y consists of finitely many ‘parallel’ infinite strips, i.e., any strip lies in a bounded neighbourhood of any other strip. As before there are two cases depending on whether these strips project properly into M_H or project to compact annuli in M_H .

Suppose that dY projects to compact annuli in M_H . The boundary lines of dY then project to circles in ∂M_H which must all lie on the same side of T , as the left annuli and right annuli of ∂M_H carry incommensurable subgroups of H . Thus we can homotop dY to arrange that dY lies entirely on one side of T . It follows that before the homotopy, one of the four intersection sets $X^{(*)} \cap Y^{(*)}$ has bounded image in M_H , so that X and Y do not cross as required.

Next suppose that dY projects to infinite strips in M_H . Each boundary line of each such strip must lie in a boundary component of M_H . As the intersection of a strip with T is compact, the two boundary lines of a single strip must lie on the same side of T , so that the strip can be homotoped to be disjoint from T . As the infinite strips forming dY are parallel, it follows that they all lie on the same side of T , and again X and Y do not cross. This completes the proof that $\Phi(T)$ is $\mathbb{Z}$ -canonical.

Let Σ_1 and Σ_2 denote the Seifert fibre spaces on each side of T , remembering that it is possible that Σ_1 equals Σ_2 . If Σ_i has a canonical annulus of M in its frontier, this annulus will be vertical and so must carry the fibre group of Σ_i . Thus if each of Σ_1 and Σ_2 are distinct and each has a canonical annulus of M in its frontier, it is immediate that G has splittings over two incommensurable infinite cyclic subgroups of H . If Σ_1 equals Σ_2 , and has a canonical annulus of M in its frontier, this yields a splitting of G over an infinite cyclic subgroup of H and a suitable conjugate of this splitting will be over an incommensurable infinite cyclic subgroup of H . Thus in this case also, G has splittings over two incommensurable infinite cyclic subgroups of H . If Σ_i has no canonical annulus in its frontier, it must meet ∂M in a torus boundary component T' . In this case, Σ_i cannot be homeomorphic to $T \times I$, as this would imply that T was inessential, so it follows that Σ_i contains an essential vertical annulus with boundary in T' . As before, whether or not Σ_1 and Σ_2 are distinct, this implies that G has splittings over two incommensurable infinite cyclic subgroups of H , as required. This completes the proof of part (1) of Theorem 2.14.

2) Let σ be a $\mathbb{Z}$ -canonical splitting of G over a subgroup H which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, such that G has splittings over two incommensurable infinite cyclic subgroups of H . As usual, we consider the cover M_H of M , whose boundary must consist of

at most one torus, some planes and some left annuli and right annuli. If G admits a splitting over an infinite cyclic subgroup L , it follows that there is an embedded essential annulus A in M such that A carries a subgroup of L . Thus the fact that G admits splittings over two incommensurable infinite cyclic subgroups of H , implies that M contains essential annuli A and B which also carry incommensurable subgroups of H . In particular, they must lift to annuli in M_H . If M_H has a torus boundary component T , then any loop in ∂M_H is homotopic into T . It follows that M_H contains essential annuli A' and B' (which may not be lifts of A and B) which carry incommensurable subgroups of H and each have one boundary component in T . But this implies that M_H is homeomorphic to $T \times I$, and hence that H has finite index in G , which contradicts the hypothesis that G splits over H . This contradiction shows that M_H does not have a torus boundary component. It follows that ∂M_H has both left annuli and right annuli, and the left and right annuli of ∂M_H carry incommensurable subgroups of H .

Let P denote an ending submanifold for the given $\mathbb{Z}$ -canonical splitting σ of G , where P is chosen as in Lemma 2.7. As the left and right annulus components of ∂M_H carry incommensurable subgroups of H , it follows that no component of dP can be a mixed annulus. Now we use the same argument as in the second part of the proof of Lemma 2.8 to show that as σ is $\mathbb{Z}$ -canonical, dP cannot have a component which is a left annulus nor a right annulus. It follows that dP consists of a single essential torus T in M_H , and the argument of Lemma 2.9 shows that T can be chosen to project to an embedding in M . Finally, the fact that ∂M_H has both left annuli and right annuli implies that T is a special canonical torus as required. This completes the proof of part (2) of Theorem 2.14. $\square$

We can summarise the results of this section as follows.

Theorem 2.15. *Let M be an orientable Haken 3-manifold with incompressible boundary, and let Φ denote the natural map from the set of isotopy classes of embedded essential annuli and tori in M to the set of splittings of $G = \pi_1(M)$ over a subgroup isomorphic to $\mathbb{Z}$ or to $\mathbb{Z} \times \mathbb{Z}$. Then*

- (1) Φ induces a bijection between the non-special canonical annuli and tori in M and the canonical splittings of G , and
- (2) Φ induces a bijection between the special canonical tori in M and those $\mathbb{Z}$ -canonical splittings of G over a subgroup H which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, such that G has splittings over two incommensurable infinite cyclic subgroups of H .

3. THE DEFORMATION THEOREM

In this section, we will give our proof of Johannson's Deformation Theorem based on the preceding work in this paper. We will give a brief comparison of our arguments with the previous proofs after our proof of Theorem 3.5.

The results so far imply the following:

Theorem 3.1. *Let M and N be orientable Haken 3-manifolds with incompressible boundary, and let Γ_M and Γ_N denote the graphs of groups structures for $\pi_1(M)$ and $\pi_1(N)$ determined by the canonical annuli and tori in M and N respectively. If $f : M \rightarrow N$ is a homotopy equivalence, the induced isomorphism $f_* : \pi_1(M) \rightarrow \pi_1(N)$ yields a graph of groups isomorphism $\Gamma_M \rightarrow \Gamma_N$.*

Proof. Let G denote $\pi_1(M)$. Theorem 2.15 tells us that there is a natural bijection between the collection of isotopy classes of all canonical annuli and tori in M and a certain collection of conjugacy classes of splittings of G , some canonical and some $\mathbb{Z}$ -canonical. As this collection of splittings of G is defined purely algebraically, it follows that the conjugacy classes of splittings of G obtained from M and from N are the same. Now Theorem 1.8 implies that they yield isomorphic graphs of groups structures for G , as required. $\square$

Before we start discussing our proof of the Deformation Theorem, we need to consider more carefully the pieces obtained by splitting M along its JSJ-system $\mathcal{F}(M)$. These pieces M_j may have compressible boundary. However, if we denote $M_j \cap \partial M$ by $\partial_0 M_j$ and denote the frontier $fr M_j$ by $\partial_1 M_j$, then each of $\partial_0 M_j$ and $\partial_1 M_j$ is incompressible in M and $\partial M_j = \partial_0 M_j \cup \partial_1 M_j$. In what follows, we will consider a triple $(L, \partial_0 L, \partial_1 L)$, where L is a compact orientable irreducible 3-manifold whose boundary is divided into subsurfaces $\partial_0 L$ and $\partial_1 L$ (which need not be connected) such that $\partial_0 L$ is incompressible in L . Often $\partial_1 L$ will also be incompressible. We will say that the pair $(L, \partial_0 L)$ is *simple* if every embedded incompressible annulus or torus in $(L, \partial_0 L)$ is parallel into $\partial_0 L$ or into $\partial_1 L$. This is the commonly used definition but it is different from the one used by Johannson in [10]. (His definition is in terms of characteristic submanifolds of Haken manifolds with boundary patterns.) We will say that $(L, \partial_0 L)$ is *fibred* if either L is an I -bundle over a surface F such that $\partial_1 L$ is the restriction of this I -bundle to ∂F , or if L is Seifert fibred and $\partial_0 L$ consists of tori and vertical annuli in ∂L .

We will want to use the JSJ-system $\mathcal{F}(M)$ of M rather than the characteristic submanifold $V(M)$. Thus the following characterisation of $\mathcal{F}(M)$ will be very useful.

Theorem 3.2. *Let M denote an orientable Haken 3-manifold with incompressible boundary, and let $S_1, \dots, S_m$ be a family of disjoint essential annuli and tori embedded in M . Let $(M_j, \partial_0 M_j, \partial_1 M_j)$ denote the manifolds obtained by splitting M along the S_i . This family is the JSJ-system $\mathcal{F}(M)$ of M if and only if*

- (1) *each $(M_j, \partial_0 M_j)$ is either simple or fibred,*
- (2) *the system $S_1, \dots, S_m$ is minimal with respect to the property (1).*

This theorem is proved towards the end of [9]. Their development uses hierarchies, and a definition of the characteristic submanifold which is different from the above characterisation, and they prove the Annulus, Torus and Enclosing Theorems in the process. In [21], Scott takes the above properties to define the JSJ-system, proves strong versions of the Annulus and Torus Theorems and then deduces the Enclosing Theorem.

Recall that in Proposition 1.2, we described how to obtain $\mathcal{F}(M)$ from $V(M)$. Now we will describe how to obtain $V(M)$ from the JSJ-system $\mathcal{F}(M)$. First we will construct a new family $\mathcal{F}'$ of essential annuli and tori in M which will be the frontier of the characteristic submanifold $V(M)$. In order to see the need for this, suppose that M consists of two Seifert fibre spaces glued together along a boundary torus T so that their fibrations do not match. Then $\mathcal{F}(M)$ consists of T , but $V(M)$ has two components homeomorphic to the two constituent Seifert fibre spaces of M . Thus the frontier of $V(M)$ consists of two copies of T , and the two components of $V(M)$ are separated by the product region between these two tori. Also if M consists of two annulus free hyperbolic manifolds glued together along a boundary

torus T , then T is the only essential torus in M and M has no essential annuli. Thus $\mathcal{F}(M)$ consists of T , and $V(M)$ consists of a product neighbourhood of T . Again the frontier of $V(M)$ consists of two copies of T . Similar examples occur when one glues manifolds along an annulus.

We construct $\mathcal{F}'$ as follows. If both sides of S_k are fibred, we add a second parallel copy of S_k , thus creating an extra complementary component X_k , which is homeomorphic to $S_k \times I$ and $\partial_1 X_k$ consists of these two copies of S_k . If both sides of S_k are simple and not fibred, we also add a second parallel copy of S_k . Splitting M along $\mathcal{F}'$ yields a manifold which is homeomorphic to the result of splitting M along $\mathcal{F}$ apart from some extra components X_k . It remains to assign the components of M split along $\mathcal{F}'$ to be part of $V(M)$ or part of $\overline{M - V(M)}$. If both sides of S_k were fibred, we assign X_k to $\overline{M - V(M)}$. If both sides of S_k were simple and not fibred, we assign X_k to $V(M)$. The other components are assigned to $V(M)$ if they are fibred and to $\overline{M - V(M)}$ otherwise. It is clear that no two adjacent pieces are assigned to $V(M)$ or to $\overline{M - V(M)}$. It follows that if a piece $(M_j, \partial_0 M_j)$ of $\overline{M - V(M)}$ is fibred, then it must be one of the extra pieces X_k described above. Thus $(M_j, \partial_0 M_j)$ is homeomorphic to $(S^1 \times S^1 \times I, \emptyset)$ or to $(S^1 \times I \times I, S^1 \times \partial I \times I)$.

We will need to use the following important result of Waldhausen.

Theorem 3.3 (Waldhausen's Homeomorphism Theorem). *If*

$$f : (M, \partial M) \rightarrow (N, \partial N)$$

is a homotopy equivalence of Haken manifolds which induces a homeomorphism on ∂M , then f can be deformed relative to ∂M to a homeomorphism.

We will note the following extension due independently to Evans [3], Swarup [25] and Tucker [27]. This will only be used to clear up some loose ends.

Theorem 3.4. *If $f : M \rightarrow N$ is a proper map of Haken manifolds inducing an injection of fundamental groups, then there is a proper homotopy $f_t : M \rightarrow N$, with $f_0 = f$, such that one of the following holds:*

- (1) f_1 is a covering map,
- (2) M is an I -bundle over a closed surface and $f_1(M) \subset \partial N$,
- (3) N (and hence also M) is a solid torus or a solid Klein bottle and f_1 is a branched covering with branch set a circle,
- (4) M is a (possibly non-orientable) handlebody and $f_1(M) \subset \partial N$.

If B and C are components of ∂M and ∂N respectively such that $f|B : B \rightarrow C$ is a covering map, then we can choose $f_t|B = f|B$, for all t .

Now we can state and prove Johannson's Deformation Theorem.

Theorem 3.5 (Johannson's Deformation Theorem). *Let M and N be orientable Haken 3-manifolds with incompressible boundary, and let $V(M)$ denote the characteristic submanifold of M . If $f : M \rightarrow N$ is a homotopy equivalence, then f is homotopic to a map g such that $g(V(M)) \subset V(N)$, the restriction of g to $V(M)$ is a homotopy equivalence from $V(M)$ to $V(N)$, and the restriction of g to $\overline{M - V(M)}$ is a homeomorphism onto $\overline{N - V(N)}$.*

Proof. We start by applying Theorem 3.1. Let $\mathcal{F}(M)$ denote the JSJ-system of M . It follows immediately from Theorem 3.1 that we can homotop f to a map g

such that g maps $\mathcal{F}(M)$ to $\mathcal{F}(N)$ by a homeomorphism and $g^{-1}\mathcal{F}(N) = \mathcal{F}(M)$. Further, it is now automatic that g induces a bijection between the components of $M - \mathcal{F}(M)$ and of $N - \mathcal{F}(N)$, and that if X is the closure of a component of $M - \mathcal{F}(M)$, and if Y denotes the closure of the component of $N - \mathcal{F}(N)$ which contains $g(X)$, then $g|X : X \rightarrow Y$ is a homotopy equivalence. If X does not meet ∂M , then $g|X : X \rightarrow Y$ is proper, i.e., $g(\partial X) \subset \partial Y$, and so Waldhausen's Homeomorphism Theorem implies that it is homotopic to a homeomorphism by a homotopy fixed on ∂X . Note that this is true whether or not X is fibred, so that the restriction of g to interior components of $V(M)$ must also be a homeomorphism.

In order to complete the proof of the Deformation Theorem, it remains to handle those components of $M - \mathcal{F}(M)$ which are not fibred and meet ∂M . Let M_j be the closure of such a component, and let N_j denote the closure of the component of $N - \mathcal{F}(N)$ which contains $g(M_j)$. Let g_j denote the induced map $M_j \rightarrow N_j$ and recall that g_j restricts to a homeomorphism of $\partial_1 M_j$ with $\partial_1 N_j$. It was shown in [16] that any simple pair $(M_j, \partial_0 M_j)$ such that $(M_j, \partial_1 M_j)$ contains an embedded π_1 -injective annulus not parallel into $\partial_1 M_j$ is fibred. In particular, it follows that no component of $\partial_0 M_j$ is an annulus. Now Theorem 3.10, which we prove below, tells us that g_j can be homotoped to a homeomorphism while keeping $\partial_1 M_j$ mapped into $\partial_1 N_j$ during the homotopy. Note that the final homeomorphism may flip the I -factor of some of the annuli in $\partial_1 M_j$. $\square$

At this point, we can give the promised comparison of the proofs of the Deformation Theorem. Johansson proved his result by directly considering the given homotopy equivalence $f : M \rightarrow N$ and describing a sequence of homotopies whose end result is g . In [26], Swarup gave a very simple proof of the special case of the Deformation Theorem when M is annulus free. (See also Remark 3.7 below.) This was an interesting application of the theory of ends. In [7], Jaco gave an alternative proof of the Deformation Theorem which was stimulated by Swarup's ideas. But he was unable to find a direct generalisation of Swarup's result on ends. As in the argument outlined above, but using completely different methods, Jaco's first step was to homotop f to a map g such that g maps $\mathcal{F}(M)$ to $\mathcal{F}(N)$ by a homeomorphism and $g^{-1}\mathcal{F}(N) = \mathcal{F}(M)$, and his second step was to deal with the non-fibred pieces of M which meet ∂M . A special case of Theorem 3.10, sufficient for this second step, was proved by Jaco in [7], but our proof is rather different, and we correct some omissions in his argument.

Now we return to the last part of the proof of Theorem 3.5 which handles those components of $M - \mathcal{F}(M)$ which are not fibred and meet ∂M . The first step in this argument is to deform $\partial_0 M_j$ into ∂N_j . We prove the following which is far more general than is required for this particular problem, but seems to be of independent interest.

Lemma 3.6. *Let $(M, \partial_0 M, \partial_1 M)$, $(N, \partial_0 N, \partial_1 N)$ be orientable 3-manifolds with M and N Haken and $\partial_0 M$ incompressible. Let $f : M \rightarrow N$ be any homotopy equivalence such that the restriction $f|_{\partial_1 M}$ is a homeomorphism onto $\partial_1 N$. Let t be a component of $\partial_0 M$ and suppose that no π_1 -injective map of an annulus into $(M, \partial_0 M)$ is essential in $(M, \partial M)$. Then $f|t$ can be deformed relative to ∂t into ∂N .*

Remark 3.7. In [26], Swarup proved a result which is the special case of this lemma when $\partial_1 M$ is empty and so t is closed. He used it to give a simple proof of Johansson's Deformation Theorem in the special case when M does not admit any essential annulus. It has been a problem for many years to find the 'correct' generalisation of Swarup's result to the case of surfaces with boundary. This lemma seems to be that generalisation.

Proof. Let $p : M_t \rightarrow M$ be the cover of M corresponding to the image of $\pi_1(t)$ in $\pi_1(M)$ and $q : N_t \rightarrow N$ the cover of N corresponding to the image of $f_*\pi_1(t)$. We have $f_t : M_t \rightarrow N_t$, a lift of f . We denote by $\tilde{t}$, a lift of t at the base point of M_t , so that $\tilde{t}$ is homeomorphic to t , and observe that it is enough to deform $f_t|_{\tilde{t}}$ into ∂N_t . We denote by $\partial_0 M_t$ the inverse image of $\partial_0 M$ under p . We next consider the map induced by inclusion on homology groups with integer coefficients:

$$i_* : H_2(M_t, \partial_0 M_t) \rightarrow H_2(M_t, \partial M_t).$$

We claim that i_* is the zero map. If $\partial_1 M_t$ is empty, the above assertion is equivalent to showing that $H_2(M_t, \partial M_t) = 0$, the case that was considered in [26]. In the general case, the proof proceeds in a similar fashion. Consider a component s of $\partial_0 M_t$ other than $\tilde{t}$. If $\pi_1(s)$ is non-trivial, we have a homotopy annulus joining any loop in s to a loop in $\tilde{t}$. By our assumption any such annulus can be homotoped relative to its boundary into ∂M_t . It follows that any loop in s is peripheral in s and thus s is either $S^1 \times I$ or $S^1 \times I$ with a closed subset of $S^1 \times 1$ removed. Moreover, there is an annulus in ∂M_t joining any non-trivial loop in s to a boundary component of $\tilde{t}$.

We next observe that $H_2(M_t, \partial_0 M_t)$ is generated by properly embedded, two-sided surfaces in $(M_t, \partial_0 M_t)$. If k is such a surface, consider a component l of ∂k which is not in $\tilde{t}$. Let s denote the component of $\partial_0 M_t$ which contains l . If l is contractible in s , it must bound a 2-disc in s which we add to k and then push off s . If l is essential in s , we add to k an annulus a in ∂M which joins l to a peripheral loop l' in $\tilde{t}$ and push $a - l'$ off ∂M_t . Thus, in either case, we can modify k without changing its image in $H_2(M_t, \partial M_t)$ so as to remove the component l . By repeating this process, we see that the image of $H_2(M_t, \partial_0 M_t)$ in $H_2(M_t, \partial M_t)$ is generated by surfaces k whose boundary is in $\tilde{t}$. Now any such surface is homologous to an incompressible surface, and we observe that any incompressible surface in M_t with boundary in $\tilde{t}$ is parallel into $\tilde{t}$. Thus the image of $H_2(M_t, \partial_0 M_t)$ in $H_2(M_t, \partial M_t)$ is zero.

Using Poincaré duality for cohomology with compact supports, it follows that the induced map in cohomology from $H_c^1(M_t, \partial_1 M_t)$ to $H_c^1(M_t)$ is zero. As f is a homotopy equivalence, the induced map $f_t : M_t \rightarrow N_t$ is a proper homotopy equivalence, so that it induces an isomorphism $H_c^1(N_t) \rightarrow H_c^1(M_t)$. Further, the hypotheses imply that f_t induces a homeomorphism on $\partial_1 M_t$, so that it also induces an isomorphism $H_c^1(N_t, \partial_1 N_t) \rightarrow H_c^1(M_t, \partial_1 M_t)$. Applying Poincaré duality in N_t , we see that the induced map in homology (with integer coefficients),

$$j_* : H_2(N_t, \partial_0 N_t) \rightarrow H_2(N_t, \partial N_t),$$

is also the zero map. Next consider $f_t(\tilde{t})$. Since f_t induces an isomorphism $\pi_1(\tilde{t}) \rightarrow \pi_1(N_t)$ and is already an embedding on $\partial \tilde{t}$, we can homotop $f_t|_{\tilde{t}}$ relative to $\partial \tilde{t}$ to become an embedding. One method to prove this is to choose a metric on N_t which blows up away from $f_t(\tilde{t})$, and homotop $f_t|_{\tilde{t}}$ to be of least area. A more classical

argument involves taking a regular neighborhood W of $f_t(\tilde{t})$, and compressing its frontier frW by adding 2-handles or by cutting W along a compressing disc for frW . One can homotop $f_t \mid \tilde{t}$ so as to avoid these compressing discs, so that one obtains a submanifold W' of N_t which contains $f_t(\tilde{t})$ and has incompressible frontier. We let t' denote a component of frW' . As f_t induces an isomorphism $\pi_1(\tilde{t}) \rightarrow \pi_1(N_t)$, it follows that there is a homotopy inverse map $N_t \rightarrow \tilde{t}$. This map will induce a proper map $t' \rightarrow \tilde{t}$ which induces an injection of fundamental groups and an injection of boundaries. It follows that this map is homotopic to a homeomorphism rel $\partial t'$. Hence $f_t \mid \tilde{t}$ can be homotoped to an embedding with image t' as required. Since $f_t(\tilde{t})$, which is now embedded, represents the zero element in $H_2(N_t, \partial N_t)$, we see that $f_t \mid \tilde{t}$ is parallel into ∂N_t . Thus $f_t \mid \tilde{t}$ can be deformed into ∂N_t relative to $\partial \tilde{t}$ and hence $f \mid t$ can be deformed into ∂N relative to ∂t . This completes the proof of Lemma 3.6. $\square$

Remark 3.8. The condition that $i^* : H_c^1(M_t, \partial_1 M_t) \rightarrow H_c^1(M_t)$ is zero can be formulated in algebraic terms. In the case when $\partial_1 M_t = \emptyset$, the condition becomes that $H_c^1(M_t) = 0$ which is equivalent to $H^1(G, \mathbb{Z}[G/H]) = 0$, where $G = \pi_1(M)$ and H is the image of $\pi_1(t)$ in $G = \pi_1(M)$. In the general case, the condition is that the induced map

$$H^1(G, H_1, \dots, H_m; \mathbb{Z}[G/H]) \rightarrow H^1(G, \mathbb{Z}[G/H])$$

is zero, where H_i represent the subgroups of G corresponding to the components of $\partial_1 M$. The condition $H^1(G, \mathbb{Z}[G/H]) = 0$ implies that the number of ends of the pair (G, H) is one. This last implication was observed by Jaco in [7].

Before continuing we need to know more about when the existence of a singular essential annulus in a 3-manifold implies the existence of an embedded one. The following, which is the Relative Annulus Theorem in [21], provides the information we need.

Theorem 3.9 (Relative Annulus Theorem). *Let $(M, \partial_0 M)$ be an orientable Haken manifold with each component of $\partial_0 M$ incompressible. Let $\alpha : (A, \partial A) \rightarrow (M, \partial_0 M)$ be a π_1 -injective map of an annulus which cannot be homotoped relative to ∂A into ∂M . Then, either*

- (1) *there is an embedded annulus in $(M, \partial_0 M)$ which is not parallel into ∂M , or*
- (2) *$\partial_0 M$ is the disjoint union of some annuli and a surface T such that (M, T) is an I -bundle. Moreover, the only (M, T) in this case which do not satisfy (1) are the product bundle over the twice punctured disc and the twisted I -bundle over the once punctured Moebius band.*

Now we are ready to prove Theorem 3.10 which we used at the end of our proof of Johannson's Deformation Theorem. Jaco stated a special case of this result as Lemma X.23 of [7]. The special case was that he assumed that $\partial_1 M$ consists of annuli and tori. Of course, this is sufficient for the application to prove the Deformation Theorem. However, there are some omissions in his proof. The main problem is that the statement of the theorem has the assumption that any embedded incompressible annulus in $(M, \partial_0 M)$ is parallel into $\partial_0 M$ or $\partial_1 M$, whereas in his proof he assumes that any π_1 -injective map of the annulus into $(M, \partial_0 M)$ is homotopic into $\partial_0 M$ or $\partial_1 M$. These two conditions are not equivalent, but Theorem 3.9 above tells us that there are a very small number of exceptional cases. We give our own proof of Jaco's result below. In the non-exceptional cases of Lemma

3.9, we replace the first part of Jaco's argument by the result of Lemma 3.6. In the exceptional cases, neither his argument nor that result apply. We give a special argument for these cases.

Theorem 3.10. *Let $(M, \partial_0 M, \partial_1 M)$, $(N, \partial_0 N, \partial_1 N)$ be orientable 3-manifolds with M and N Haken, and $\partial_0 M$ incompressible. Suppose that no component of $\partial_0 M$ or $\partial_0 N$ is an annulus and that any embedded incompressible annulus in $(M, \partial_0 M)$ is parallel into $\partial_0 M$ or $\partial_1 M$. If $f : M \rightarrow N$ is a homotopy equivalence such that the restriction $f|_{\partial_1 M}$ is a homeomorphism onto $\partial_1 N$, then f can be deformed to a homeomorphism while keeping $\partial_1 M$ mapped into $\partial_1 N$ during the homotopy.*

Proof. We start by considering a component t of $\partial_0 M$, and observing that either Lemma 3.6 applies, or we are in the special case (2) in Theorem 3.9.

First suppose that the special case of Theorem 3.9 does not arise, so that any π_1 -injective annulus in $(M, \partial_0 M)$ is properly homotopic into ∂M . Then Lemma 3.6 applies to every component of $\partial_0 M$, so we can homotop f rel $\partial_1 M$ to a proper map. We will continue to call this map f .

Next we claim that, for each component t of $\partial_0 M$, we can deform $f|_t$ to an embedding so that during the homotopy each component of ∂t either stays fixed or is moved across an annulus component of $\partial_1 N$. The argument here uses the fact that $f|_t$ is π_1 -injective, but this is not enough. It is also necessary to use the fact that any π_1 -injective annulus in $(M, \partial_0 M)$ is properly homotopic into ∂M . This is another omission in Jaco's proof. We let Σ denote the component of ∂N which contains $f(t)$.

If t is closed, then $f|_t$ is homotopic to a covering map of Σ of some degree d . Theorem 1.3 of [19] tells us that, as M and N are orientable, d must equal 1, so that we have deformed $f|_t$ to a homeomorphism onto its image, as required.

Next we consider the case when t has non-empty boundary. We already know that $f|_t$ embeds ∂t in Σ . Now we consider the lift f_t of $f|_t$ into the cover Σ_t of Σ whose fundamental group equals $f_*\pi_1(t)$. We can homotop f_t rel ∂t to an embedding. This homotopy induces a homotopy of $f|_t$. We will now assume that this homotopy has been done and use the same notation for the new maps. Thus f_t is an embedding in Σ_t , and we denote its image by X . Let $\tilde{\Sigma}$ denote the universal cover of Σ , and let $\tilde{X}$ denote the pre-image in $\tilde{\Sigma}$ of X . The full pre-image in $\tilde{\Sigma}$ of t consists of $\tilde{X}$ and all its translates by $\pi_1(\Sigma)$. Suppose that $f|_t$ is not an embedding. Then some translate $g\tilde{X}$ of $\tilde{X}$ must meet $\tilde{X}$ but not equal $\tilde{X}$. Let $\tilde{Y}$ denote a component of this intersection, and let Y denote the image of $\tilde{Y}$ in Σ_t . This will be a compact subsurface of X . Note that the boundary of Y consists of circles which project to ∂t . Consider an essential (possibly singular) loop C in Y and let l denote a line in $\tilde{Y}$ above C . Let α denote a generator of the stabiliser of l . Then α lies in the stabiliser H of $\tilde{X}$ and in the stabiliser gHg^{-1} of $g\tilde{X}$. Thus there is β in H such that $\alpha = g\beta g^{-1}$. As $g\tilde{X}$ is not equal to $\tilde{X}$, we know that g does not lie in H . Thus we obtain a π_1 -injective (possibly singular) annulus A in M with both ends on t and one end at C , such that A is not properly homotopic into t . But any π_1 -injective annulus in $(M, \partial_0 M)$ is properly homotopic into ∂M . It follows that some component D of the closure of $\partial M - t$ is an annulus and that A can be homotoped to cover D , keeping ∂A in t during the homotopy. As D is a union of components of $\partial_0 M$ and $\partial_1 M$, and as no component of $\partial_0 M$ is an annulus, it follows that D is a component of $\partial_1 M$. We let E denote the component $f(D)$ of $\partial_1 N$. A

particular consequence of the preceding argument is that C must be homotopic in t into a component of ∂t . Hence every essential loop on Y is homotopic into a component of ∂X , so that Y must be an annulus parallel to a component of ∂X . As the boundary of Y consists of circles which project to ∂t , it follows that Y must project to E . In this case, we homotop the two parallel components of ∂t across E . Note that this reduces the number of components of $f^{-1}(f(\partial t)) \cap t$ by 2. Repeating this as needed yields the required homotopy of $f|_t$ to an embedding such that during the homotopy each component of ∂t either stays fixed or is moved across an annulus component of $\partial_1 N$.

After doing this for all components of $\partial_0 M$, if $f|_{\partial M}$ fails to be a homeomorphism, we must have two components t_1 and t_2 of $\partial_0 M$ whose images intersect or even coincide, or we must have components t_0 of $\partial_0 M$ and t_1 of $\partial_1 M$ whose images coincide. In the first case, let S denote a component of the intersection of t_1 and t_2 . (Note that t_1 and t_2 may both be closed, in which case so is S .) Then S is bounded by some components of $\partial_0 N \cap \partial_1 N$, so that S must be a union of certain components S_i of $\partial_0 N$ and $\partial_1 N$. As in the previous paragraph, any essential loop C in S_i yields an annulus in $(M, \partial_0 M)$, joining t_1 and t_2 , which then implies that ∂S_i is not empty and that C is homotopic into ∂S_i . This implies that each S_i must be an annulus. As no component of $\partial_0 N$ is an annulus, it follows that S must be an annulus component of $\partial_1 N$. Thus we can change f by flip homotopies on these annuli and obtain a homeomorphism from ∂M to ∂N . Finally Waldhausen's Homeomorphism Theorem shows that we can properly homotop f to a homeomorphism, completing the proof in this case. In the second case, we have components t_0 of $\partial_0 M$ and t_1 of $\partial_1 M$ whose images coincide. Hence t_0 and t_1 are homotopic in M . If t_0 and t_1 are closed, this implies that M is homeomorphic to $t_0 \times I$. Otherwise, there is a submanifold W of M homeomorphic to $t_0 \times I$, such that $t_0 \times \partial I$ corresponds to the union of t_0 and t_1 . If C denotes any component of ∂t_0 and A denotes the annulus in ∂W joining C to a component of ∂t_1 , then A must be parallel into ∂M . It follows that M is homeomorphic to $t_0 \times I$ in this case also. In either case, it is easy to homotop f to a homeomorphism keeping $\partial_1 M$ mapped into $\partial_1 N$ during the homotopy. This completes the proof of Theorem 3.10, so long as the special case (2) in Theorem 3.9 never occurs.

Now suppose that for some component t of $\partial_0 M$, we are in special case (2) of Theorem 3.9. As no component of $\partial_0 M$ is an annulus, we see that in this special case, $(M, \partial_0 M)$ is an I -bundle, whose base surface is the twice punctured disc or the once punctured Moebius band. As $\pi_1(M)$ is free of rank 2, so is $\pi_1(N)$, so that each of M and N is a handlebody of genus 2.

Case 3.11. $(M, \partial_0 M)$ is $(\Sigma \times I, \Sigma \times \partial I)$ where Σ denotes the twice punctured disc.

Thus $\partial_1 M$ consists of three annuli. As f induces a homeomorphism $\partial_1 M \rightarrow \partial_1 N$, it follows that $\partial_1 N$ also consists of three annuli. Choosing one boundary component from each of these three annuli in ∂N gives us three disjoint simple closed curves in ∂N . Each must be essential in N , and no two can be homotopic in N , for that would yield an annulus in M joining boundary circles of distinct components of $\partial_1 M$. As 3 is the maximum number of disjoint essential simple closed curves one can have on the closed orientable surface of genus 2, it follows that $\partial_0 N$ must consist of two pairs of pants, or equivalently, twice punctured discs. Now let S denote one of the two components of $\partial_0 N$. The given map $f : (M, \partial_1 M) \rightarrow (N, \partial_1 N)$ has a homotopy inverse $g : (N, \partial_1 N) \rightarrow (M, \partial_1 M)$, and we consider the composite map

$\varphi : S \subset N \xrightarrow{g} M \rightarrow \Sigma$, where the last map is simply projection. Then $\varphi : S \rightarrow \Sigma$ is proper and π_1 -injective and so is properly homotopic to a covering map, as S is not an annulus. As S and Σ each have Euler number equal to -1 , it follows that $\varphi : S \rightarrow \Sigma$ is homotopic to a homeomorphism rel ∂S . Thus we can homotop f to a homeomorphism while keeping $\partial_1 M$ mapped into $\partial_1 N$ during the homotopy, as required.

Case 3.12. $(M, \partial_0 M)$ is $(\Sigma \tilde{\times} I, \Sigma \tilde{\times} \partial I)$ where Σ denotes the once punctured Moebius band.

In this case, $\partial_0 M$ is a connected double cover of Σ , and $\partial_1 M$ consists of two annuli. As before, we let S denote a component of $\partial_0 N$. Recall that S is not an annulus. Now let $g : (N, \partial_1 N) \rightarrow (M, \partial_1 M)$ denote a homotopy inverse to f , and consider the composite map $\varphi : S \subset N \xrightarrow{g} M \rightarrow \Sigma$, where the last map is simply projection. Then $\varphi : S \rightarrow \Sigma$ is proper and π_1 -injective and so is properly homotopic to a covering map, as S is not an annulus. The degree of this covering must be even, as S is orientable and Σ is not. As S is an incompressible subsurface of ∂N , we have $\chi(S) \geq \chi(\partial N) = -2$. As $\chi(\Sigma) = -1$, it follows that the covering has degree 2. It follows that each of S and $\partial_0 M$ is the orientable double cover of Σ . Hence we can homotop f to a homeomorphism while keeping $\partial_1 M$ mapped into $\partial_1 N$ during the homotopy, as required. $\square$

We complement the above result by observing what happens if we drop the assumption that no component of $\partial_0 M$ or $\partial_0 N$ is an annulus.

Theorem 3.13. *Let $(M, \partial_0 M, \partial_1 M)$, $(N, \partial_0 N, \partial_1 N)$ be orientable 3-manifolds with M and N Haken, and $\partial_0 M$ incompressible. Suppose that any embedded incompressible annulus in $(M, \partial_0 M)$ is parallel into $\partial_0 M$ or $\partial_1 M$ and that M and N are not solid tori. If $f : M \rightarrow N$ is a homotopy equivalence such that the restriction $f|_{\partial_1 M}$ is a homeomorphism onto $\partial_1 N$, then f can be deformed to a homeomorphism while keeping $\partial_1 M$ mapped into $\partial_1 N$ during the homotopy.*

Proof. As in the proof of Theorem 3.10, we can homotop f to be proper while keeping $\partial_1 M$ mapped into $\partial_1 N$ during the homotopy. Now we apply Theorem 3.4. We conclude that f is properly homotopic to a map f_1 which is a homeomorphism unless a) M is an I -bundle over a closed surface and $f_1(M) \subset \partial N$, or b) N (and hence also M) is a solid torus and f_1 is a branched covering with branch set a circle, or c) M is a handlebody and $f_1(M) \subset \partial N$. If f_1 is a homeomorphism, we can alter f_1 and the homotopy so as to arrange that $\partial_1 M$ is mapped into $\partial_1 N$ during the homotopy, which completes the proof of the theorem in this case. Case b) is excluded in the hypotheses of our theorem, so it remains to eliminate the two cases a) and c).

Suppose that we have case a), so that M is an I -bundle over a closed surface F and $f_1(M)$ lies in a component S of ∂N . As f is a homotopy equivalence, the inclusion of S in N must also be a homotopy equivalence, so that N is homeomorphic to $S \times I$. If M is the trivial I -bundle over F , then f must map each component of ∂M to S with degree 1. Otherwise, the bundle is non-trivial and f maps ∂M to S with degree 2. In either case, M contains embedded annuli which are not properly homotopic into ∂M , so $\partial_1 M$ must be non-empty. Let Y be a component of $\partial_1 N$ in S and X be a component of $\partial_1 M$ which maps to Y by a homeomorphism. Let Σ denote a component of ∂Y , and let C denote the component of ∂X mapped to Σ .

Whether M is a trivial or non-trivial I -bundle, there is an annulus A embedded in M with one end equal to C which cannot be properly homotoped into ∂M . Let D denote the other end. Suppose that D meets a component Z of $\partial_1 M$. As f maps $\partial_1 M$ to $\partial_1 N$ by a homeomorphism, $f(Z)$ is disjoint from Y . As $f(D)$ is homotopic to $f(C) = \Sigma$, we see that $f(D)$ can be homotoped out of $f(Z)$, and hence that D can be homotoped out of Z . By repeating this argument for any other components of $\partial_1 M$ which meet D , we can arrange that D lies in $\partial_0 M$. But this immediately contradicts the condition that any embedded incompressible annulus in $(M, \partial_0 M)$ is parallel into $\partial_0 M$ or $\partial_1 M$. This contradiction shows that case a) cannot occur.

Finally suppose that we have case c), so that M , and hence N , is a handlebody. If t denotes a component of $\partial_0 M$, the argument in the proof of Theorem 3.10 shows that either $f|t$ can be modified rel ∂t to be an embedding, or that the image of t contains an annulus S which is a union of components of $\partial_0 M$ and $\partial_1 M$. If we regard S as an annulus in $(M, \partial_0 M)$, the assumptions of our theorem imply that S is parallel into $\partial_0 M$ or $\partial_1 M$. The first case is only possible if t is an annulus, and the second is only possible if S is a component of $\partial_1 M$. In the second case, we can further homotop t to an embedding by moving two components of ∂t across S . Thus either, we can homotop $f|t$ to an embedding or t is an annulus and the boundary of N is a torus consisting of the image of t . As N is a handlebody, it must be a solid torus, contradicting our assumptions. Thus $f|t$ can be homotoped to an embedding relative to its boundary, for each component t of $\partial_0 M$, and we complete the argument as in Theorem 3.10. $\square$

REFERENCES

- [1] B. H. Bowditch, Cut points and canonical splittings of hyperbolic groups, *Acta Math.* 180 (1998), no. 2, 145–186. MR **99g**:20069
- [2] M. J. Dunwoody and M. Sageev, JSJ splittings for finitely presented groups over slender groups, *Invent. Math.* 135(1999), 25–44. MR **2000b**:20050
- [3] B. Evans, Boundary respecting maps of 3-manifolds, *Pacific J. Math.* 42 (1972), 639–655. MR **47**:9626
- [4] M. H. Freedman, J. Hass and P. Scott, Least area incompressible surfaces in 3-manifolds, *Invent. Math.* 71 (1983), 609–642. MR **85e**:57012
- [5] F. Fujiwara and P. Papasoglu, JSJ-Decompositions of finitely presented groups and complexes of groups, Preprint (1998).
- [6] W. Heil, On P^2 -irreducible 3-manifolds, *Bull. Amer. Math. Soc.* 75(1969), 772–775. MR **40**:4958
- [7] W. Jaco, *Lectures on three-manifold topology*, Amer. Math. Soc., Providence, R.I., 1980. MR **81k**:57009
- [8] W. Jaco and J. H. Rubinstein, PL minimal surfaces in 3-manifolds, *J. Differential Geom.* 27(1988), no.3, 493–524. MR **89e**:57009
- [9] W. Jaco and P. B. Shalen, Seifert fibered spaces in 3-manifolds, *Memoirs of Amer. Math. Soc.*, vol. 21, Number 220 (1979). MR **81c**:57010
- [10] K. Johannson, Homotopy equivalences of 3-manifolds with boundaries, *Lecture Notes in Mathematics* 761, Springer-Verlag, 1979. MR **82c**:57005
- [11] P. K. Kim and J. L. Tollefson, PL involutions of fibered 3-manifolds, *Trans. Amer. Math. Soc.* 232 (1977), 221–237. MR **56**:13223
- [12] W. H. Meeks and P. Scott, Finite group actions on 3-manifolds, *Invent. Math.* 86 (1986), 287–346. MR **88b**:57039
- [13] W. H. Meeks, III and S.-T. Yau, The classical Plateau problem and the topology of three dimensional manifolds, *Topology* 21 (1982), 409–442. MR **84g**:53016
- [14] B. Leeb and P. Scott, A geometric characteristic splitting in all dimensions, *Comm. Math. Helv.* 75 (2000), 201–215. CMP 2000:16

- [15] N. Nakauchi, On free boundary Plateau problem for general-dimensional surfaces, Osaka J. Math. 21 (1984), 831-841. MR **86i**:49041
- [16] W. D. Neumann and G.A. Swarup, Canonical decompositions of 3-manifolds, Geometry and Topology, 1 (1997), 21-40. MR **98k**:57033
- [17] E. Rips and Z. Sela, Cyclic splittings of finitely generated groups and the canonical JSJ decomposition, Ann. of Math. 146 (1997), 53-109. MR **98m**:20044
- [18] R. Schoen and S.-T. Yau, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature, Ann. of Math. (2) 110 (1979), 127-142. MR **81k**:58029
- [19] P. Scott, On sufficiently large 3-manifolds, Quart. J. Math. Oxford Ser. (2) 23 (1972), 159-172; correction, *ibid.* (2) 24 (1973), 527-529. MR **52**:4295
- [20] P. Scott, A new proof of the annulus and torus theorems, American J. of Math. 102 (1980), 241-277. MR **81f**:57006
- [21] P. Scott, Strong annulus and torus theorems and the enclosing property of characteristic submanifolds, Quarterly J. of Math. Oxford (2), 35 (1984), 485-506. MR **86i**:57016
- [22] P. Scott, The symmetry of intersection numbers in group theory, Geometry and Topology 2(1998), 11-29, Correction (*ibid.*) (1998). MR **99k**:20076a; MR **99k**:20076b
- [23] P. Scott and G. A. Swarup, Splittings of groups and intersection numbers, Geometry and Topology 4 (2000), 179-218. CMP 2000:16
- [24] Z. Sela, Structure and rigidity in (Gromov) hyperbolic groups and rank 1 Lie groups, Geom. Funct. Anal. 7 (1997), 561-593. MR **98j**:20044
- [25] G. A. Swarup, Boundary preserving maps of 3-manifolds, Proc. Amer. Math. Soc. 78 (1980), no. 2, 291-294. MR **81g**:57008
- [26] G. A. Swarup, On a theorem of Johansson, J. London Math. Soc. (2) (1978), no. 3, 560-562. MR **80c**:57007
- [27] T. W. Tucker, Boundary-reducible 3-manifolds and Waldhausen's theorem, Michigan Math. J. 20 (1973), 321-327. MR **48**:12537
- [28] F. Waldhausen, On the determination of some bounded 3-manifolds by their fundamental group alone, Proc. of the Internat. Symp. on Topology and its Applications, Herceg-Nov, Yugoslavia, Beograd 1969, 331-332.
- [29] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math., 87 (1968), 56-88. MR **36**:3366

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